# Secondary Products, By-products and the Commodity Technology Assumption

Elio Londero

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## Abstract

A more general method for preparing commodity by commodity input-output tables under the commodity technology assumption is presented for the case when there are produced and non-produced by-products, originating in both principal and secondary production. Existing methods of the same family are shown to be special cases of the one presented. A numerical example (not included) shows that incorrect model specification is a sufficient condition for unwarranted negative coefficients. Finally, the model is shown to satisfy desirable properties of an input-output system.



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# SECONDARY PRODUCTS, BY-PRODUCTS, AND THE COMMODITY TECHNOLOGY ASSUMPTION

By *Elio Londero* *

## 1. Introduction

In discussing the preparation of input-output tables, Chenery and Clark (1959, p. 139) suggested the method, "followed to a considerable degree in Japan," of deducting the value of secondary production and its required inputs from the values of the industry's production and inputs, respectively. To that effect, the authors proposed using "as a guide" the input structure of the industry where those secondary products are principal, "since the basic input data collected from the establishments are not usually broken down on a product basis." Adhering strictly to the commodity technology assumption, Van Rijckeghem (1967) proposed a method for the preparation of commodity by commodity input-output tables under the commodity technology assumption when there is only principal and secondary production. If there is only one technique to produce a commodity and each secondary product is principal to one industry, data could be arranged according to principal products in a square absorption or use matrix where industries are assumed to be linear combinations of principal and secondary production activities. Then, the solution of a simple set of linear equations provides the commodity by commodity matrix.

More recently, ten Raa et al. (1984) extended the method to include the existence of by-products that are produced at the margin by separate production processes and result only from principal production. Londero (1990) extended the approach to include by-products originated in principal production that are not produced at the margin by other activities. This type of approach is generalized in section 2 of this paper to allow for both types of by-products originating in both principal and secondary production. Then, in section 3 all above mentioned methods are shown to be special cases of the one presented here. Section 4 explores the consequences of incorrect model specification when the data do correspond to the commodity technology assumption. Finally, in section 5 the model is proven to satisfy some desirable properties expected from input-output systems.

When input-output tables are use for analytical purposes, particularly those exploring the effects of changes in demand on production and primary-input use, it is important to distinguish between those goods that are produced at the margin and those that are not. A good will be said to be produced at the margin, or simply produced, if a change in the demand for it is met fully by changes in its domestic production; it will be said to be nonproduced at the margin, or simply nonproduced, if a change in its demand does not result in changes in its domestic production.[1] Goods will be considered either produced or nonproduced for simplicity's sake, and in line with preceding works on the subject.

An industry is conceived as an aggregation of production processes. The production process that generates the maximum value of production in the industry is said to be principal, and its marginally produced outputs called principal outputs. The remaining production processes are called secondary, and the corresponding marginally produced outputs are called secondary outputs. The commodity technology assumption postulates a biunique correspondence between produced outputs and production processes.

An output *b* will be called a by-product of a principal or secondary production process *j* when a change in the demand for *b* does not affect the output level of *j*, while an increase in the demand for one or more of the others does. The by-product, in turn, may be produced at the margin by a *separate* production process, or it may be nonproduced at the margin for the system as a whole. Nonproduced by-products used as current inputs by other production processes would be nonproduced inputs of those activities.

A final word of caution is required, to avoid possible confusion in terminology. It is the view followed in this paper that the classification of outputs into principal, secondary or by-products, should be distinguished from the association of the cost structure to the commodity or to the industry. The commodity technology assumption, as understood here, refers to commodities that are *produced* by activities, as opposed to by-products obtained from principal or from secondary production. The methods proposed by ten Raa et al. (1984), by Londero

(1990), and that of this paper, all assume that industries are simple aggregations of activities or production processes (cost compositions), each uniquely characterized by its output, and that there is only one activity for each of those outputs. From this perspective, they all correspond to the commodity technology assumption. This view differs from that of Kop Jansen and ten Raa (1990), who called the method proposed by ten Raa et al. (1984) a "mixed technology model" because they saw it as a mix of the commodity and the by-product technology assumptions.

## 2. A more general approach

The relationship between inputs used, and outputs obtained may be presented as

$$[1]\,\mathbf{U} + [1]\,\mathbf{W} = [1]\,\mathbf{X} \tag{1}$$

where [1] is a unitary vector, $\mathbf{U} = [U_{ij}]$ is the matrix containing the value of produced input $i$ used to obtain the value of total industry output, $\mathbf{W} = [W_{hj}]$ is that for nonproduced inputs and the value added $h$, and $\mathbf{X}$ is the transpose of the make matrix. This matrix may be expressed as

$$\mathbf{X} = \begin{bmatrix} {}^{\mathrm{p}}\hat{\mathbf{X}} + {}^{\mathrm{bp}}\mathbf{X} + {}^{\mathrm{s}}\mathbf{X} + {}^{\mathrm{bs}}\mathbf{X} \\ \hdashline {}^{\mathrm{bp}}\mathbf{X}_{\mathrm{h}} + {}^{\mathrm{bs}}\mathbf{X}_{\mathrm{h}} \end{bmatrix} \tag{2}$$

where the dotted line indicates partitioning, the circumflex accent indicates a diagonal matrix, ${}^{\mathrm{p}}\hat{\mathbf{X}} = [{}^{p}X_{jj}]$ contains the principal production of industry $j$, ${}^{\mathrm{bp}}\mathbf{X} = [{}^{bp}X_{ij}]$ contains by-products $i$ originated in the principal production of $j$ and produced as principal by other industries, ${}^{\mathrm{bp}}\mathbf{X}_{\mathrm{h}} = [{}^{bp}X_{hj}]$ contains nonproduced by-products $h$ originated in the production of $j$, ${}^{\mathrm{s}}\mathbf{X} = [{}^{s}X_{ij}]$ contains secondary products $i$ of industry $j$, ${}^{\mathrm{bs}}\mathbf{X} = [\Sigma_k\, {}^{bs}X_{ikj}]$ contains produced by-products $i$ originated in secondary production of $k$ by industry $j$, and, finally, ${}^{\mathrm{bs}}\mathbf{X}_{\mathrm{h}} = [\Sigma_k\, {}^{bs}X_{hkj}]$ contains nonproduced by-products $h$ of secondary production $k$ by industry $j$. Matrix $\mathbf{X}$ will have dimensions $(n + t) \times n$, where $n$ are the produced commodities and $t$ are the nonproduced by-products.

It should be noted that $U_{ij}$ and $W_{hj}$ have been defined in terms of marginally produced inputs and marginally nonproduced inputs and value added, instead of simply in terms of inputs and value added. Inputs of one industry that are only nonproduced by-products of others have been removed from $\mathbf{U} = [U_{ij}]$ and assigned to $\mathbf{W} = [W_{hj}]$. References to value added will henceforth be omitted for the sake of brevity.

It should also be noted that ${}^{bp}X_{jj} = 0$ and ${}^{s}X_{jj} = 0$. An output would not be principal and simultaneously a by-product or a secondary output of the same production process. However, it is conceivable for ${}^{bs}X_{jj}$ to be positive and for that reason total industry output of $j$, $X_{jj}$, is denoted differently from total principal output of $j$, ${}^{p}X_{jj}$.

From (1) and (2), the relationship between inputs used and outputs obtained may be expressed as

$$[1]\,\mathbf{U} + [1]\,\mathbf{W} = [1]\,[{}^{\mathrm{p}}\hat{\mathbf{X}} + {}^{\mathrm{bp}}\mathbf{X} + {}^{\mathrm{s}}\mathbf{X} + {}^{\mathrm{bs}}\mathbf{X}] + [1]\,[{}^{\mathrm{bp}}\mathbf{X}_{\mathrm{h}} + {}^{\mathrm{bs}}\mathbf{X}_{\mathrm{h}}] \tag{3}$$

Following Londero (1990), define matrix $\overline{\mathbf{W}}$ as matrix $\mathbf{W}$ augmented by as many zero rows as nonproduced by-products that, while not specified in $\mathbf{W}$, are present in ${}^{\mathrm{bp}}\mathbf{X}_{\mathrm{h}}$ or in ${}^{\mathrm{bs}}\mathbf{X}_{\mathrm{h}}$ (i.e., by-products that are exclusively final goods). Similarly, ${}^{\mathrm{bp}}\overline{\mathbf{X}}_{\mathrm{h}}$ and ${}^{\mathrm{bs}}\overline{\mathbf{X}}_{\mathrm{h}}$ will be matrices ${}^{\mathrm{bp}}\mathbf{X}_{\mathrm{h}}$ and ${}^{\mathrm{bs}}\mathbf{X}_{\mathrm{h}}$ augmented by as many zero rows as nonproduced inputs different from nonproduced by-products (e.g., labor) are present in $\mathbf{W}$. Matrices $\overline{\mathbf{W}}$, ${}^{\mathrm{bp}}\overline{\mathbf{X}}_{\mathrm{h}}$, and ${}^{\mathrm{bs}}\overline{\mathbf{X}}_{\mathrm{h}}$ would then have the same dimensions with rows and columns corresponding to the same nonproduced goods. Thus, the relation between inputs and outputs may be written

$$[1]\,\mathbf{U} + [1]\,\overline{\mathbf{W}} = [1]\,[{}^{\mathrm{p}}\hat{\mathbf{X}} + {}^{\mathrm{bp}}\mathbf{X} + {}^{\mathrm{s}}\mathbf{X} + {}^{\mathrm{bs}}\mathbf{X}] + [1]\,[{}^{\mathrm{bp}}\overline{\mathbf{X}}_{\mathrm{h}} + {}^{\mathrm{bs}}\overline{\mathbf{X}}_{\mathrm{h}}] \tag{4}$$

If $\mathbf{A} = [a_{ij}]$ is the $n \times n$ commodity by commodity coefficient matrix for the produced goods and $\mathbf{F} = [f_{hj}]$ the corresponding $m \times n$ matrix for the nonproduced ones, then from (4) it follows that

$$\begin{aligned} \mathbf{A}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] &= \mathbf{U} - {}^{bp}\mathbf{X} - {}^{bs}\mathbf{X} \\ \mathbf{F}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] &= \overline{\mathbf{W}} - {}^{bp}\overline{\mathbf{X}}_h - {}^{bs}\overline{\mathbf{X}}_h \end{aligned} \tag{5}$$

From (5) the coefficient matrices can be derived as

$$\begin{aligned} \mathbf{A} &= [\mathbf{U} - {}^{bp}\mathbf{X} - {}^{bs}\mathbf{X}]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \\ \mathbf{F} &= [\overline{\mathbf{W}} - {}^{bp}\overline{\mathbf{X}}_h - {}^{bs}\overline{\mathbf{X}}_h]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \end{aligned} \tag{6}$$

which are general expressions for obtaining commodity by commodity matrices when there is secondary production, as well as produced and nonproduced by-products originating in both principal and secondary production.[2]

## 3. Other methods as special cases

Results obtained by other authors that followed the same type of approach can be shown to be special cases of equations (6). When there are no by-products of any kind, i.e., when ${}^{bp}\mathbf{X} = {}^{bs}\mathbf{X} = {}^{bp}\overline{\mathbf{X}}_h = {}^{bs}\overline{\mathbf{X}}_h = [0]$, equations (6) provide the result obtained early by Van Rijckeghem (1967):

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \\ \mathbf{F} &= \mathbf{W}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \end{aligned} \tag{7}$$

When by-products originate only in principal production and are all produced by separate production processes, i.e., when ${}^{bs}\mathbf{X} = [0]$ and ${}^{bp}\overline{\mathbf{X}}_h = {}^{bs}\overline{\mathbf{X}}_h = [0]$, equations (6) simplify to

$$\begin{aligned} \mathbf{A} &= [\mathbf{U} - {}^{bp}\mathbf{X}]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \\ \mathbf{F} &= \mathbf{W}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \end{aligned} \tag{8}$$

This is the result obtained by ten Raa et al. (1984).[3]

Finally, when produced and nonproduced by-products originate only in principal production, i.e., when only ${}^{bs}\mathbf{X} = [0]$ and ${}^{bs}\mathbf{X}_h = [0]$, equations (6) simplify to the result obtained by Londero (1990):

$$
\begin{aligned}
\mathbf{A} &= [\mathbf{U} - {}^{bp}\mathbf{X}]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \\
\mathbf{F} &= [\bar{\mathbf{W}} - {}^{bp}\bar{\mathbf{X}}_{h}]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1}
\end{aligned}
\qquad (9)
$$

## 4. Some implications of alternative methods[4]

Van Rijckeghem's (1967) proposal assumes that only principal and secondary production exist. In practice, the use of this method implies that by-products should be classified as either principal or secondary outputs. It is logical to think that by-products will be included as part of the output of principal or secondary production processes where they originate, that is

$$
\begin{aligned}
&{}^{p}X_{jj} + \Sigma_i\, {}^{bp}X_{ij} + \Sigma_h\, {}^{bp}X_{hj} \\
&{}^{s}X_{ij} + \Sigma_i\, {}^{bs}X_{ij} + \Sigma_h\, {}^{bs}X_{hj}
\end{aligned}
\qquad (10)
$$

The results of following this classification and applying equations (6) are coefficients resulting from calculating input cost as a proportion of the value of the output *basket* generated by each activity. Further processing of the data is required to calculate the coefficients for the by-products and reassign them (with negative signs) to the corresponding matrices in order to obtain the correct **A** and **F** matrices. That processing amounts to identifying and classifying by-products in the same manner as required by the more general procedure proposed in this paper (equation (6)). Errors in the coefficients will arise if data for by-products were arranged differently than in equations (10). A likely error would be to classify produced by-products as secondary production.

When all outputs are allocated to the correct activities, Van Rijckeghem's method is expected to render nonnegative coefficients. The resulting matrix, however, would not be a commodity by commodity one, since coefficients have not been calculated with respect to marginally produced output.

The approach followed by ten Raa et al. (1984) takes into account the existence of by-products that originate only in principal production, and assumes that all of them are also produced by separate industries. Then, it is logical to assume that nonproduced by-products of principal production would be treated as principal outputs, and that by-products of secondary production would be treated as part of the corresponding secondary outputs. That implies

$$
\begin{gathered}
{}^{p}X_{jj} + \Sigma_h {}^{bp}X_{hj} \\
{}^{s}X_{ij} + \Sigma_i {}^{bs}X_{ij} + \Sigma_h {}^{bs}X_{hj}
\end{gathered}
\tag{11}
$$

A numerical example shows that if there are nonproduced by-products of principal production and by-products of secondary production, the **A** matrix resulting from applying this method may include unwarranted negative coefficients.

Finally, when produced and nonproduced by-products from principal production are taken into account, but no by-products of secondary production are allowed (Londero, 1990), the logical assumption for the classification of these by-products seems to be adding them to secondary production, i.e.,

$$
{}^{s}X_{ij} + \Sigma_i {}^{bs}X_{ij} + \Sigma_h {}^{bs}X_{hj} \tag{12}
$$

As in the case of the preceding method, when there are by-products of secondary production, this specification may also render unwarranted negative coefficients.

In sum, both the ten Raa et al. and the Londero (1990) methods result in errors in estimating some cost structures due to their inability to deal with produced by-products of secondary production, and may render unwarranted negative coefficients. The use of the more general method may reduce the incidence of unwarranted negative coefficients obtained in other studies dealing with the existence of by-products (see ten Raa, 1988).

It should be noted that when there are by-products that are produced by other activities, the general method may not meet the conditions for $[a_{ij}]$ to be nonnegative (Steenge, 1990), since these by-products would show as negative demands to the sector producing them as principals.

Therefore, careful inspection of the resulting **A** matrix is required in order to separate warranted from unwarranted negative coefficients. The analysis of unwarranted negatives, including that of the accuracy and correct classification of the data, would determine whether the data satisfies the stringent conditions of the commodity technology assumption.[5]

## 5. Desirable properties

Kop Jansen and ten Raa (1990) presented four desirable properties for an input-output model to satisfy. They called these properties price invariance, scale invariance, material balance, and financial balance. These authors showed that what they call the "mixed technology model" would violate two of those desirable properties.[6] In this section, it will be shown that the general commodity technology model does not violate any of these properties when conditions are properly stated.

*Price invariance* signifies that if price ratios $p_j$ are applied to the absorption and make matrices, the model returns value coefficients $a_{ij}\, p_i \,/\, p_j$, i.e., the original coefficients updated by the corresponding relative price changes. Consequently, from (6) the model presented in this paper will be price invariant if it verifies that

$$\hat{\mathbf{p}}\,\mathbf{A}\,\hat{\mathbf{p}}^{-1} = [\hat{\mathbf{p}}\,\mathbf{U} - \hat{\mathbf{p}}\,({}^{bp}\mathbf{X} + {}^{bs}\mathbf{X})]\,[\hat{\mathbf{p}}\,({}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X})]^{-1} \qquad (13)$$

where $\hat{\mathbf{p}}$ is a diagonal matrix containing the price ratios $p_i$. It can be easily proven that this is the case.[7]

*Scale invariance* means that if inputs and outputs corresponding to one industry are multiplied by a scalar $s$, the model returns the same final coefficients. From (6) this condition can be stated formally as

$$\mathbf{A} = [\mathbf{U}\,\hat{\mathbf{s}} - ({}^{bp}\mathbf{X} - {}^{bs}\mathbf{X})\,\hat{\mathbf{s}}]\,[({}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X})\,\hat{\mathbf{s}}]^{-1} \qquad (14)$$

It can also be easily proven that it is satisfied as well.[8]

The third condition is to satisfy Leontief's *material balance*. To formulate the condition, a distinction needs to be made between the accounting identities in terms of total inputs and total outputs, and the identities required from the analytical model expressed in terms of produced and nonproduced outputs. From this second perspective, Leontief's material balance may be stated by saying that total supply of a certain produced commodity must be equal to total intermediate consumption plus total final use. To state the condition, express produced by-products $i$ of activity $j$ as a proportion $b_{ij}$ of principal output of activity $j$, i.e.,

$$ {}^{bp}\mathbf{X} + {}^{bs}\mathbf{X} = \mathbf{B}\,({}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}) \tag{15} $$

Then, Leontief's material balance will be[9]

$$ {}^{\ell}\mathbf{x} + \mathbf{B}\,{}^{\ell}\mathbf{x} = {}^{u}\mathbf{A}\,{}^{\ell}\mathbf{x} + \mathbf{d} $$

where ${}^{\ell}\mathbf{x}$ is the column vector of total *produced* output arranged by commodity, i.e., ${}^{\ell}\mathbf{x} = ({}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X})\,[1]$, ${}^{u}\mathbf{A}$ is a matrix of input-output coefficients showing the *use* of produced input $i$ per unit value of production of output $j$, and $\mathbf{d}$ is the vector of final use of produced outputs. Consequently, Leontief's material balance may also be expressed as

$$ {}^{\ell}\mathbf{x} = ({}^{u}\mathbf{A} - \mathbf{B})\,{}^{\ell}\mathbf{x} + \mathbf{d} \tag{16} $$

To verify compliance with this condition, replace (15) in (6) and rearrange in order to obtain

$$ \mathbf{A} + \mathbf{B} = \mathbf{U}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1} \tag{17} $$

where $\mathbf{U}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1}$ is matrix ${}^{u}\mathbf{A}$ in equation (16).[10] From (17), rearranging, postmultiplying by [1], and replacing by ${}^{\ell}\mathbf{x} = ({}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X})\,[1]$, it follows that

$$ (\mathbf{A} + \mathbf{B})\,{}^{\ell}\mathbf{x} = \mathbf{U}\,[1] \tag{18} $$

Since **d** is defined as produced output not used as an input

$$\mathbf{d} = {}^{\ell}\mathbf{x} + \mathbf{B}\,{}^{\ell}\mathbf{x} - \mathbf{U}\,[1]$$

it follows that

$$\mathbf{U}\,[1] = {}^{\ell}\mathbf{x} + \mathbf{B}\,{}^{\ell}\mathbf{x} - \mathbf{d} \tag{19}$$

Replacing (19) into (18) and rearranging yields

$${}^{\ell}\mathbf{x} = (\mathbf{A} + \mathbf{B})\,{}^{\ell}\mathbf{x} - \mathbf{B}\,{}^{\ell}\mathbf{x} + \mathbf{d} \tag{20}$$

Equation (20) is Leontief's material balance condition. It can be proven by replacing ${}^{u}\mathbf{A} = (\mathbf{A} + \mathbf{B})$ from (17) into (20), thus obtaining equation (16).

Therefore, the method proposed here does comply with the material balance condition when Leontief's material balance is stated taking produced by-products into account. Matrix **A** is that of the coefficients of additional *production* of inputs required per unit value of produced output ${}^{\ell}\mathbf{x}$ when there are produced by-products. In other words, the traditional equation for Leontief's material balance ($\mathbf{x} = {}^{u}\mathbf{A}\,\mathbf{x} + \mathbf{d}$) is a particular case of equation (20) in which $\mathbf{B} = [0]$. Equation (20) can be used to calculate total production requirements of a given bill of produced final goods as long as marginal coefficients are equal to average coefficients:

$${}^{\ell}\mathbf{x} = [\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{d} \tag{21}$$

where **A** is obtained from equation (6), i.e., coefficients $a_{ij}$ are net of produced by-products.

The model is also asked to satisfy a fourth condition called the *financial balance*. It may be stated by saying that the total value of produced outputs by an industry should equal the sum of all produced inputs used by the industry, plus all nonproduced inputs and value added

$$[1]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] = [1]\,{}^{u}\mathbf{A}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] + [1]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] - [1]\,\mathbf{U} \tag{22}$$

Then, replacing by ${}^{u}\mathbf{A} = (\mathbf{A} + \mathbf{B})$ it should be proven that

$$[1]\,(\mathbf{A} + \mathbf{B})\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] + [1]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] - [1]\,\mathbf{U} = [1]\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] \tag{23}$$

or, simplifying, that

$$[1]\,(\mathbf{A} + \mathbf{B})\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}] = [1]\,\mathbf{U} \tag{24}$$

which has been proven to be the case in (17).

It has been shown that the general model complies with all four conditions when they are stated in terms of produced inputs and outputs. Therefore, particular specifications of the model --e.g., those proposed by ten Raa et al. (1984) and Londero (1990)-- will also comply with said conditions.[11]

## 6. Conclusions

A general method for modeling data that correspond to the commodity technology assumption has been proposed. This method takes into account the existence of produced and nonproduced by-products originating in principal and in secondary production. It is general in the sense that previously proposed methods have been proven here to be special cases of it that result from restrictive assumptions regarding the existence of secondary output and by-products.

Using a method that does not allow for proper treatment of all by-products is a sufficient condition for obtaining unwarranted negative coefficients. Since the omission of produced and nonproduced by-products by other methods leads to the misclassification of outputs, their use may result in negative coefficients even when the data do correspond to the commodity technology assumption. Therefore, the incidence of negatives found in earlier studies may be reduced by using this method.

The general method alone does not avoid the risk of misclassifying outputs -- e.g., produced by-products as secondary production--, but its use may force a more careful look at the characteristics of the production processes that underlie the data. This practical aspect highlights the importance of exogenous information. The preparation of commodity by commodity matrices demands a considerable effort in classifying outputs. That effort may require help from the engineering field, as well as specific information requests during the data-gathering process. In the words of Stone (1960, pp. 35-6), "... the best course is to seek information from the industries concerned since they should be able to provide better answers than any mechanical procedure for disentangling costs and sales."

Finally, it was proven that the method complies with the desirable properties of price and scale invariance, as well as with material and financial balance conditions when they are properly stated in terms of produced inputs and outputs.

## Footnotes

* *Inter-American Development Bank.* Opinions expressed in this paper are those of the author and do not intend to represent views of the Bank. Comments by Thijs ten Raa, Rob Vos and two anonymous referees, and valuable help from Pablo Londero are gratefully acknowledged. The author remains solely responsible for the remaining errors.

1. For a more detailed treatment see Londero (1992, 1996). Note that changes in the composition of demand for the output mix of the activity may change the nonproduced (produced) characteristic of an individual output.

2. A longer paper with a more detailed derivation of equation (6) and a numerical example is available from the author upon request.

3. Note that in deriving (7) and (8), since ${}^{bp}\overline{\mathbf{X}}_h = {}^{bs}\overline{\mathbf{X}}_h = [0]$, $\overline{\mathbf{W}}$ becomes $\mathbf{W}$. Actually, ten Raa et al. (1984) did not present the equation to obtain $\mathbf{F}$, but (8) is the equation implied by their method.

4. This section reviews some implications of applying the alternative methods described in the preceding section. These implications derive in part from a simple and detailed numerical example, excluded for space reasons, but available upon request.

5. Rainer and Richter (1992) provide an analysis of the negative coefficients obtained in calculating a commodity by commodity matrix.

6. Kop Jansen and ten Raa (1990) reviewed the performance of several models, including those analyzed in this paper.

7. Recalling that $(\mathbf{Y}\,\mathbf{Z})^{-1} = \mathbf{Z}^{-1}\,\mathbf{Y}^{-1}$, equation (13) becomes $\hat{\mathbf{p}}\,[\mathbf{U} - ({}^{bp}\mathbf{X} + {}^{bs}\mathbf{X})][({}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X})]^{-1}\,\hat{\mathbf{p}}^{-1}$.

8. Similarly, equation (14) becomes $[\mathbf{U} - ({}^{bp}\mathbf{X} - {}^{bs}\mathbf{X})]\,\hat{\mathbf{s}}\,\hat{\mathbf{s}}^{-1}[\ ({}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X})]^{-1}$.

9. The author would like to thank Thijs ten Raa for pointing out an error in the original formulation of this equation.

10. Since ${}^{u}a_{ij}$ is the use of input $i$ per unit value of produced output $j$, ${}^{u}\mathbf{A}\,\hat{\mathbf{X}} + {}^{u}\mathbf{A}\,{}^{s}\mathbf{X} = \mathbf{U}$, from where ${}^{u}\mathbf{A} = \mathbf{U}\,[{}^{p}\hat{\mathbf{X}} + {}^{s}\mathbf{X}]^{-1}$.

11. Cf. Kop Jansen and ten Raa (1990). Note that these particular cases satisfy the logical conditions, but the data may not correspond to the assumptions implicit in each one of them.

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